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Representations of commutative asynchronous automata

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ABSTRACT

Ito (1976, 1978) [14,17] provided representations of strongly connected automata by group-matrix type automata. This shows the close connection between strongly connected automata with their automorphism groups. In this paper we deal with commutative asynchronous automata. In particular, we introduce and study normal commutative asynchronous automata and cyclic commutative asynchronous automata. Some properties on endomorphism monoids of these automata are given. Also, the representations of normal commutative asynchronous automata and cyclic commutative asynchronous automata are provided by S-automata and regular S-automata, respectively. The cartesian composition $\mathbf{A} \circ \mathbf{B}$ of a strongly connected automaton $\mathbf{A}$ and a cyclic commutative asynchronous automaton $\mathbf{B}$ is studied. It is shown that the endomorphism monoid $E(\mathbf{A} \circ \mathbf{B})$ of automaton $\mathbf{A} \circ \mathbf{B}$ is a Clifford monoid. Finally, a representation of $\mathbf{A} \circ \mathbf{B}$ is provided by regular Clifford monoid matrix-type automaton. This generalizes and extends the representations of strongly connected automata given by Ito (1976) [14].

1. Introduction and preliminaries

Automata considered in this paper will be always automata without outputs. That is to say, an *automaton* $\mathbf{A} = (A, X, \delta)$ consists of the following data:

(i) A is a finite nonempty set, called a *state set*;
(ii) X is a finite nonempty set, called an *alphabet*;
(iii) δ is a function, called a *state transition function* from $A \times X$ into A.

Let X^* (X^+) denote the *free monoid* (*free semigroup*) generated by X. An element of X^* is called a *word* over X and ε is called the *empty word*. The state transition function can be extended to the function from $A \times X^*$ to A by

(i) $\delta(a, \varepsilon) = a$ for any $a \in A$;
(ii) $\delta(a, xu) = \delta(\delta(a, x), u)$ for any $a \in A$, $x \in X$ and any $u \in X^*$.

Let $\mathbf{A} = (A, X, \delta)$ and $\mathbf{B} = (B, X, \gamma)$ be automata and let ρ be a mapping from A into B. If $\rho(\delta(a, x)) = \gamma(\rho(a), x)$ holds for any $a \in A$ and any $x \in X$, then ρ is called a *homomorphism* from $\mathbf{A}$ into $\mathbf{B}$. If a homomorphism ρ is bijective, then ρ is called an *isomorphism*. If there exists an isomorphism from $\mathbf{A}$ onto $\mathbf{B}$, then $\mathbf{A}$ and $\mathbf{B}$ are said to be *isomorphic* to each

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other and denoted by $\mathbf{A} \cong \mathbf{B}$. Moreover, a homomorphism (an isomorphism) from $\mathbf{A}$ into itself is called an *endomorphism* (an *automorphism*) of $\mathbf{A}$. It is clear that $E(\mathbf{A})$ $(G(\mathbf{A}))$ of all endomorphisms (automorphisms) of $\mathbf{A}$ forms a monoid (group) on the usually composition, called the *endomorphism monoid* (*automorphism group*) of $\mathbf{A}$.

The study of automorphism groups and endomorphism monoids of automata was started by [6] and [24] and followed by [7,25,1,23,4,3,2,22,19,5].

Recall that an automaton $\mathbf{A} = (A, X, \delta)$ is said to be *connected* if for any $a, b \in A$ there exist a sequence $a = a_0, a_1, \dots, a_n = b$ of states and a sequence $x_0, x_1, \dots, x_n$ of letters such that $\delta(a_{i-1}, x_i) = a_i$ or $\delta(a_i, x_i) = a_{i-1}$ for $i = 1, 2, \dots, n$. Moreover, if for any $a, b \in A$ there exists a word $u \in X^*$ such that $\delta(a, u) = b$, then $\mathbf{A}$ is said to be *strongly connected*. Fleck [6] proved that if $\mathbf{A} = (A, X, \delta)$ is a strongly connected automaton, then $E(\mathbf{A}) = G(\mathbf{A})$ and $|G(\mathbf{A})|$ divides $|A|$ (for more results on automorphism groups of strongly connected automata see [6,7]). Ito [14] introduced and studied *group-matrix type automata* of order n on a group G. It is shown that for any strongly connected automaton $\mathbf{A} = (A, X, \delta)$, there exists a group-matrix type automaton $\mathbf{A}' = (\widehat{G(\mathbf{A})_n}, X, \delta_\Psi)$ of order n on automorphism group $G(\mathbf{A})$ such that $\mathbf{A}' \cong \mathbf{A}$ (see [14–17] or [18], for more details). This gives representations of strongly connected automata by group-matrix type automata.

Following Fleck and Ito, we are going to introduce and study normal commutative asynchronous automata and S-automata in Section 2. Also, we give representations of normal commutative asynchronous automata by S-automata. In Section 3 we study cyclic commutative asynchronous automata. It is shown that cyclic commutative asynchronous automata are normal. Also, we give representations of cyclic commutative asynchronous automata by regular S-automata. We study in Section 4 the cartesian composition $\mathbf{A} \circ \mathbf{B}$ of a strongly connected automaton $\mathbf{A}$ and a cyclic commutative asynchronous automaton $\mathbf{B}$. It is shown that the endomorphism monoid $E(\mathbf{A} \circ \mathbf{B})$ of automaton $\mathbf{A} \circ \mathbf{B}$ is a Clifford monoid. As a preparation of Section 6, we introduce in Section 5 Clifford monoid matrix-type automata. Finally, we provide a representation of $\mathbf{A} \circ \mathbf{B}$ by regular Clifford monoid matrix-type automaton, in Section 6. This generalizes and extends the representations of strongly connected automata given by Ito in [14].

An automaton $\mathbf{A} = (A, X, \delta)$ is said to be *commutative* if $\delta(a, uv) = \delta(a, vu)$ for any $a \in A$ and any $u, v \in X^*$. An automaton $\mathbf{A} = (A, X, \delta)$ is an *asynchronous automaton* if $\delta(a, xx) = \delta(a, x)$ for any $a \in A$ and any $x \in X$. A *commutative asynchronous automaton* means a commutative and asynchronous automaton. For more information on commutative asynchronous automata and asynchronous automata we refer to [10–13].

The following results are basis of this paper.

Lemma 1. *Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton. Then $\delta(a, v) = \delta(a, vv)$ holds for any $a \in A$ and any $v \in X^*$.*

Proof. Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton and let $a \in A$. Clearly, $\delta(a, v) = \delta(a, vv)$ holds, if $v = \varepsilon$. Suppose now that $v = x_1 x_2 \cdots x_n$, where $x_i \in X$, $i = 1, 2, \dots, n$. Then

$$\begin{aligned}
\delta(a, v) &= \delta(a, x_1 x_2 \cdots x_n) = \delta\big(\delta(a, x_1), x_2 \cdots x_n\big) \\
&= \delta\big(\delta\big(a, x_1^2\big), x_2 \cdots x_n\big) = \delta\big(\delta\big(\delta\big(a, x_1^2\big), x_2\big), x_3 \cdots x_n\big) \\
&= \delta\big(\delta\big(\delta\big(a, x_1^2\big), x_2^2\big), x_3 \cdots x_n\big) = \delta\big(\delta\big(a, x_1^2 x_2^2\big), x_3 \cdots x_n\big) \\
&= \cdots \\
&= \delta\big(a, x_1^2 x_2^2 \cdots x_n^2\big) = \delta(a, vv). \qquad \square
\end{aligned}$$

Proposition 1. *Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton and let a, b be a pair of states in A. If $\delta(a, u) = b$ and $\delta(b, v) = a$ for some $u, v \in X^*$, then $a = b$.*

Proof. Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton and let $a, b \in A$. Suppose that $\delta(a, u) = b$ and $\delta(b, v) = a$ for some $u, v \in X^*$. Then $\delta(a, uv) = \delta(b, v) = a$. Hence, by Lemma 1 we have

$$\delta(a, v) = \delta\big(\delta(b, v), v\big) = \delta(b, v) = a.$$

Therefore, $a = \delta(a, uv) = \delta(a, vu) = \delta(\delta(a, v), u) = \delta(a, u) = b$. $\square$

Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton. For any $u \in X^*$, define a mapping λ_u from A into itself as follows:

$$(\forall a \in A) \quad \lambda_u(a) = \delta(a, u).$$

The set $\{\lambda_u \mid u \in X^*\}$ is denoted by $\Lambda(\mathbf{A})$.

Proposition 2. *Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton. Then $\Lambda(\mathbf{A})$ is a commutative idempotent submonoid of $E(\mathbf{A})$.*

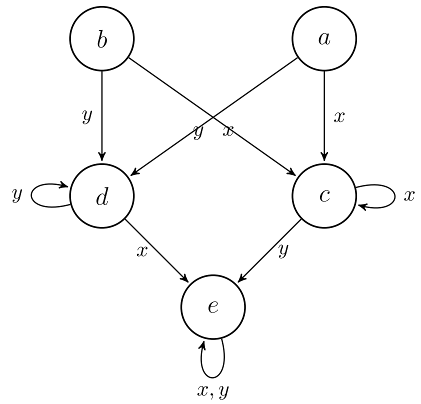

Fig. 1. State transition diagram of automaton $\mathbf{A} = (A, X, \delta)$.

Proof. Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton. It is easy to see that $\lambda_\varepsilon \in E(\mathbf{A})$. For any nonempty word $u \in X^*$ and any $x \in X$, we have

$$\begin{aligned}\lambda_u\big(\delta(a,x)\big) &= \delta\big(\delta(a,x),u\big) \quad (\textit{by the definition of } \lambda_u)\\ &= \delta(a,xu)\\ &= \delta(a,ux) \quad (\textit{since } \mathbf{A} \textit{ is commutative})\\ &= \delta\big(\lambda_u(a),x\big).\end{aligned}$$

This shows that $\lambda_u \in E(\mathbf{A})$ and hence $\Lambda(\mathbf{A}) \subseteq E(\mathbf{A})$. It is also a routine matter to verify that ${\lambda_u}^2 = \lambda_u$ and $\lambda_u\lambda_v = \lambda_v\lambda_u = \lambda_{uv} \in \Lambda(\mathbf{A})$ for any $u, v \in X^*$. Thus $\Lambda(\mathbf{A})$ is a commutative idempotent submonoid of $E(\mathbf{A})$. □

For undefined notions and notations concerning automata see [8] and [18].

2. Normal commutative asynchronous automata

In this section we shall introduce and study normal commutative asynchronous automata and S-automata. Also, we provide representations of normal commutative asynchronous automata by S-automata.

Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton. Define a binary relation $\leqslant_{\mathbf{A}}$ on A as follows:

$$(\forall a, b \in A) \quad a \leqslant_{\mathbf{A}} b \quad \Leftrightarrow \quad \delta(b, w) = a \quad \text{for some } w \in X^*.$$

Since $\delta(a, \varepsilon) = a$ holds for any $a \in A$, $\leqslant_{\mathbf{A}}$ is reflexive. It follows immediately from Proposition 1 that $\leqslant_{\mathbf{A}}$ is antisymmetric. Also, it is a routine matter to verify that $\leqslant_{\mathbf{A}}$ is transitive. Thus, $\leqslant_{\mathbf{A}}$ is a partial order on A.

If the partially ordered set $(A, \leqslant_{\mathbf{A}})$ is a meet semilattice and $a \wedge_{\mathbf{A}} b$ denotes the greatest lower bound of a and b in A, then $\wedge_{\mathbf{A}}$ is a binary operation on A and $(A, \wedge_{\mathbf{A}})$ is a commutative idempotent semigroup. Recall the translations of a semigroup [9], we have the translation ρ_a associated with each element a in A, i.e.,

$$(\forall b \in A) \quad \rho_a(b) = a \wedge_{\mathbf{A}} b.$$

The set of all ρ_a is denoted by $\Omega(\mathbf{A})$.

Definition 1. A commutative asynchronous automaton $\mathbf{A} = (A, X, \delta)$ is said to be normal if $(A, \leqslant_{\mathbf{A}})$ is a meet semilattice and $\Omega(\mathbf{A}) \subseteq E(\mathbf{A})$. The class of all normal commutative asynchronous automata is denoted by $\mathcal{NCAA}$.

The following example shows that $(A, \leqslant_{\mathbf{A}})$ may not be a meet semilattice for a commutative asynchronous automaton $\mathbf{A} = (A, X, \delta)$. Thus, the condition, $(A, \leqslant_{\mathbf{A}})$ is a meet semilattice, is needed in the above definition.

Example 1. Fig. 1 shows the state transition diagram of a commutative asynchronous automaton $\mathbf{A} = (A, X, \delta)$.

It is easy to see that the Hasse diagram (Fig. 2) of the partially ordered set $(A, \leqslant_{\mathbf{A}})$ is not a meet semilattice.

In the following we will give another example which shows that ρ_a may not be an endomorphism of a commutative asynchronous automaton $\mathbf{B} = (B, X, \delta)$, even if $(B, \leqslant_{\mathbf{B}})$ is a meet semilattice, where $a \in B$.

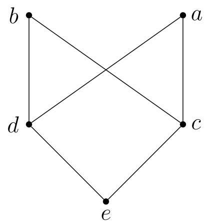

Fig. 2. Hasse diagram of $(A, \leqslant_{\mathbf{A}})$.

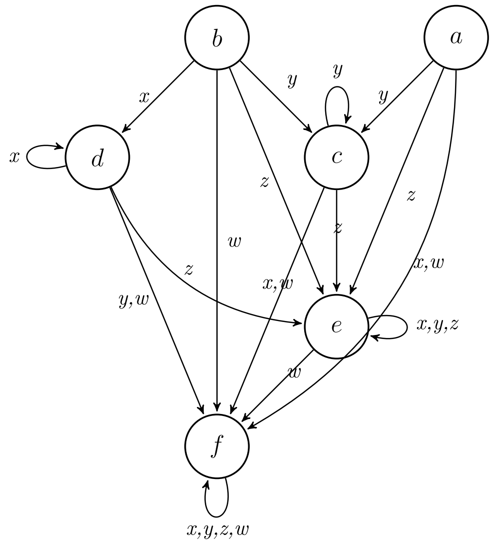

Fig. 3. State transition diagram of automaton $\mathbf{B} = (B, X, \delta)$.

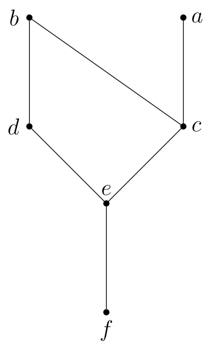

Fig. 4. Hasse diagram of $(B, \leqslant_{\mathbf{B}})$.

Example 2. Fig. 3 shows the state transition diagram of a commutative asynchronous automaton $\mathbf{B} = (B, X, \delta)$.

It is easy to see that the Hasse diagram (Fig. 4) of the partially ordered set $(B, \leqslant_{\mathbf{B}})$ is a semilattice. Also, we have

$$\rho_a\big(\delta(b, x)\big) = a \wedge_{\mathbf{A}} \delta(b, x) = a \wedge_{\mathbf{A}} d = e;$$
$$\delta\big(\rho_a(b), x\big) = \delta(a \wedge_{\mathbf{A}} b, x) = \delta(c, x) = f.$$

This implies that $\rho_a(\delta(b, x)) \neq \delta(\rho_a(b), x)$ and so ρ_a is not an endomorphism of automaton **B**.

Proposition 3. *Let $\mathbf{A} = (A, X, \delta)$ be an automaton in $\mathcal{NCAA}$ and $E(\mathbf{A})$ the endomorphism monoid of* **A**. *Then the following statements are true*:

(i) *$\Omega(\mathbf{A})$ is a commutative idempotent subsemigroup of $E(\mathbf{A})$*;
(ii) *$\Omega(\mathbf{A}) \cup \Lambda(\mathbf{A})$ is a commutative idempotent submonoid of $E(\mathbf{A})$*;
(iii) *$\Omega(\mathbf{A})$ is an ideal of $\Omega(\mathbf{A}) \cup \Lambda(\mathbf{A})$.*

Proof. Let $\mathbf{A} = (A, X, \delta) \in \mathcal{NCAA}$. We know from Theorem 1.1.2 in [9] that $\Omega(\mathbf{A})$ is a semigroup under usual composition of mappings and isomorphic to $(A, \wedge_{\mathbf{A}})$. Thus, $\Omega(\mathbf{A})$ is a commutative idempotent subsemigroup of $E(\mathbf{A})$, since $\Omega(\mathbf{A}) \subseteq E(\mathbf{A})$.

To show that $\Omega(\mathbf{A}) \cup \Lambda(\mathbf{A})$ is a commutative idempotent submonoid of $E(\mathbf{A})$ and $\Omega(\mathbf{A})$ is an ideal of $\Omega(\mathbf{A}) \cup \Lambda(\mathbf{A})$, we need only to prove that $\lambda_u \rho_a = \rho_a \lambda_u \in \Omega(\mathbf{A})$ holds for any $u \in X^*$ and any $a \in A$, since both $\Lambda(\mathbf{A})$ and $\Omega(\mathbf{A})$ are commutative idempotent subsemigroups of $E(\mathbf{A})$. It is true that $\lambda_\varepsilon \rho_a = \rho_a \lambda_\varepsilon = \rho_a$ for any $a \in A$. Suppose now that $u \in X^+$. Then for any $a, b \in A$, we have

$$\begin{aligned}
\lambda_u \rho_a(b) &= \lambda_u(a \wedge_{\mathbf{A}} b) \quad (\textit{since } \rho_a \textit{ is a translation}) \\
&= \delta(a \wedge_{\mathbf{A}} b, u) \quad (\textit{by the definition of } \lambda_x) \\
&= \delta\big(\rho_a(b), u\big) \\
&= \rho_a\big(\delta(b, u)\big) \quad (\textit{since } \rho_a \textit{ is an endomorphism}) \\
&= \rho_a \lambda_u(b)
\end{aligned}$$

and

$$\begin{aligned}
\rho_a \lambda_u(b) &= \rho_a\big(\delta(b, u)\big) \quad (\textit{by the definition of } \lambda_u) \\
&= \delta\big(\rho_a(b), u\big) \quad (\textit{since } \rho_a \textit{ is an endomorphism}) \\
&= \delta(a \wedge_{\mathbf{A}} b, u) \quad (\textit{since } \rho_a \textit{ is a translation}) \\
&= \delta\big(\rho_b(a), u\big) \quad (\textit{since } \rho_b \textit{ is a translation}) \\
&= \rho_b\big(\delta(a, u)\big) \quad (\textit{since } \rho_b \textit{ is an endomorphism}) \\
&= b \wedge_{\mathbf{A}} \big(\delta(a, u)\big) \quad (\textit{since } \rho_b \textit{ is a translation}) \\
&= \rho_{\delta(a,u)}(b) \quad (\textit{since } \rho_{\delta(a,u)} \textit{ is a translation}).
\end{aligned}$$

This shows that $\lambda_u \rho_a = \rho_a \lambda_u = \rho_{\delta(a,u)} \in \Omega(\mathbf{A})$ holds for any $u \in X^*$ and any $a \in A$, as required. □

In order to provide representations of automata in $\mathcal{NCAA}$, we introduce S-automata.

Definition 2. Let $(S, \wedge)$ be a finite commutative idempotent semigroup and let T be an ideal of S. An automaton $\mathbf{T} = (T, X, \delta_\varphi)$ is called an S-automaton, if state transition function δ_φ is defined by $\delta_\varphi(t, x) = t \wedge \varphi(x)$ for any $t \in T$ and any $x \in X$, where φ is a mapping from X into S.

Since X^+ is the free semigroup on X, the mapping φ in the above definition can be extended to a semigroup homomorphism from X^+ into S as follows:

$$\big(\forall x \in X,\ \forall u \in X^+\big) \quad \varphi(xu) = \varphi(x) \wedge \varphi(u).$$

It is easy to verify that an S-automaton is a commutative asynchronous automaton. Also, we have

Proposition 4. *Let $(S, \wedge)$ be a finite commutative idempotent semigroup and T an ideal of S. If $\mathbf{T} = (T, X, \delta_\varphi)$ is an S-automaton, then T can be embedded into the endomorphism monoid $E(\mathbf{T})$ of automaton $\mathbf{T}$.*

Proof. Let $\mathbf{T} = (T, X, \delta_\varphi)$ be an S-automaton. Define a mapping ϱ_s for any $s \in S$ as follows:

$$\varrho_s : T \to T, \quad \varrho_s(t) = s \wedge t.$$

Then for any $x \in X$, we have

$$\varrho_s\big(\delta_\varphi(t, x)\big) = \varrho_s\big(t \wedge \varphi(x)\big) = s \wedge t \wedge \varphi(x) = \delta_\varphi(s \wedge t, x) = \delta_\varphi\big(\varrho_s(t), x\big).$$

This shows that $\varrho_s \in E(\mathbf{T})$.

Now, we prove that the mapping $\phi : s \mapsto \varrho_s$ is a semigroup homomorphism from S into $E(\mathbf{T})$. For this purpose, take $u \mapsto \varrho_u$, $v \mapsto \varrho_v$. Then for any $t \in T$, we have

$$\varrho_{u \wedge v}(t) = u \wedge v \wedge t = \varrho_u\big(\varrho_v(t)\big) = \varrho_u \varrho_v(t).$$

This implies that $u \wedge v \mapsto \varrho_{u \wedge v}$ holds. Hence, the mapping ϕ is a homomorphism.

Next, we prove that $\phi|_T$ is injective. Suppose now that $t, t' \in T$ and that $\varrho_t = \varrho_{t'}$. Then

$$\begin{aligned} t &= t \wedge t = \varrho_t(t) \quad (\textit{by the definition of } \varrho_t) \\ &= \varrho_{t'}(t) \quad (\textit{since } \varrho_t = \varrho_{t'}) \\ &= t' \wedge t = t \wedge t' \\ &= \varrho_t(t') \\ &= \varrho_{t'}(t') \quad (\textit{since } \varrho_t = \varrho_{t'}) \\ &= t' \wedge t' = t'. \end{aligned}$$

This shows that $\phi|_T$ is injective. Hence, T can be embedded into the endomorphism monoid $E(\mathbf{T})$ of automaton $\mathbf{T}$. □

Let $\mathbf{A} = (A, X, \delta)$ be an automaton in $\mathcal{NCAA}$. From Proposition 2, $\Lambda(\mathbf{A}) \subseteq E(\mathbf{A})$. Define a mapping φ from X^* into $\Lambda(\mathbf{A})$:

$$(\forall u \in X^*) \quad \varphi(u) = \lambda_u.$$

We can easily verify that φ is a homomorphism from the free monoid X^* into $E(\mathbf{A})$. Furthermore, take $S = \Omega(\mathbf{A}) \cup \Lambda(\mathbf{A})$ in Definition 2. We have an S-automaton $\boldsymbol{\Omega}(\mathbf{A}) = (\Omega(\mathbf{A}), X, \delta_\varphi)$, since it follows from Proposition 3(iii) that $\Omega(\mathbf{A})$ is an ideal of $\Omega(\mathbf{A}) \cup \Lambda(\mathbf{A})$.

Define a mapping θ from A into $\Omega(\mathbf{A})$:

$$(\forall a \in A) \quad \theta(a) = \rho_a.$$

It is easy to verify that θ is bijective. Now, we prove that θ is a homomorphism from automaton $\mathbf{A}$ into automaton $\Omega(\mathbf{A})$, i.e., $\theta(\delta(a, x)) = \delta_\varphi(\theta(a), x)$ holds for any $a \in A$ and any $x \in X$. In fact, for any $b \in A$, we have

$$\begin{aligned} \big(\theta\big(\delta(a, x)\big)\big)(b) &= \rho_{\delta(a,x)}(b) \\ &= b \wedge_{\mathbf{A}} \delta(a, x) \quad (\textit{since } \rho_{\delta(a,x)} \textit{ is a translation}) \\ &= \rho_b\big(\delta(a, x)\big) \quad (\textit{since } \rho_b \textit{ is a translation}) \\ &= \delta\big(\rho_b(a), x\big) \quad (\textit{since } \rho_b \textit{ is an endomorphism}) \\ &= \delta(a \wedge_{\mathbf{A}} b, x) \end{aligned}$$

and

$$\begin{aligned} \big(\delta_\varphi\big(\theta(a), x\big)\big)(b) &= \big(\delta_\varphi(\rho_a, x)\big)(b) \quad \big(\textit{since } \theta(a) = \rho_a\big) \\ &= \big(\rho_a \circ \varphi(x)\big)(b) \quad (\textit{by Definition } 2) \\ &= (\rho_a \circ \lambda_x)(b) \quad (\textit{by the definition of } \varphi) \\ &= \rho_a\big(\delta(b, x)\big) \quad (\textit{by the definition of } \lambda_x) \\ &= \delta\big(\rho_a(b), x\big) \quad (\textit{since } \rho_a \textit{ is an endomorphism}) \\ &= \delta(a \wedge_{\mathbf{A}} b, x) \quad (\textit{since } \rho_a \textit{ is a translation}). \end{aligned}$$

This implies that $\theta(\delta(a, x)) = \delta_\varphi(\theta(a), x)$ holds for any $a \in A$ and any $x \in X$. Therefore, θ is an isomorphism from the automaton $\mathbf{A}$ onto the automaton $\Omega(\mathbf{A})$.

Thus, we have proved

Theorem 1. *Let $\mathbf{A} = (A, X, \delta)$ be an automaton in $\mathcal{NCAA}$ and let S be the commutative idempotent semigroup $\Omega(\mathbf{A}) \cup \Lambda(\mathbf{A})$. Then $\mathbf{A}$ is isomorphic to the S-automaton $(\Omega(\mathbf{A}), X, \delta_\varphi)$.*

The following lemma is obvious and its proof is omitted.

Lemma 2. *Let S and S' be two finite commutative idempotent semigroups. If S is isomorphic to S', then for any S-automaton $\mathbf{T} = (T, X, \delta_\varphi)$, there exists an S'-automaton $\mathbf{T}' = (T', X, \delta_{\varphi'})$ such that $\mathbf{T} \cong \mathbf{T}'$.*

From the above Lemma 2 and Theorem 1, we immediately have

Corollary 1. *Let $\mathbf{A}$ be an automaton in $\mathcal{NCAA}$ and let S be a finite commutative idempotent semigroup such that $S \cong \Omega(\mathbf{A}) \cup \Lambda(\mathbf{A})$. Then $\mathbf{A}$ is isomorphic to some S-automaton.*

3. Cyclic commutative asynchronous automata

In this section we will introduce and study cyclic commutative asynchronous automata and regular S-automata. It is shown that every cyclic commutative asynchronous automaton is normal. Also, the representations of cyclic commutative asynchronous automata are provided by regular S-automata.

Let $\mathbf{A} = (A, X, \delta)$ be an automaton. Recall that a state g in A is called a *generator* of $\mathbf{A}$ [19] if for any $a \in A$ there exists $u \in X^*$ such that $\delta(g, u) = a$. The set of all generators of $\mathbf{A}$ is denoted by $Gen(\mathbf{A})$. An automaton $\mathbf{A}$ is said to be *cyclic* if $Gen(\mathbf{A}) \neq \emptyset$. A *cyclic commutative asynchronous automaton* means a cyclic and commutative asynchronous automaton. The class of all cyclic commutative asynchronous automata is denoted by $\mathcal{CCAA}$.

Let $\mathbf{A} = (A, X, \delta)$ be an automaton in $\mathcal{CCAA}$. It is true that $\mathbf{A}$ has a unique generator. In fact, if g, h are generators of $\mathbf{A}$, then there exist $u, v \in X^*$ such that $\delta(g, u) = h$ and $\delta(h, v) = g$. Thus, it follows from Proposition 1 that $g = h$.

We have shown

Lemma 3. *If $\mathbf{A}$ is an automaton in $\mathcal{CCAA}$, then $\mathbf{A}$ has a unique generator.*

In the following we will give some properties of an endomorphism of automaton $\mathbf{A}$ in $\mathcal{CCAA}$.

Proposition 5. *Let $\mathbf{A} = (A, X, \delta)$ be an automaton in $\mathcal{CCAA}$ and g the unique generator of $\mathbf{A}$. If ρ is an endomorphism of $\mathbf{A}$, then the following statements are true.*

(i) *If $\rho(g) = \sigma(g)$ for some $\sigma \in E(\mathbf{A})$, then $\rho = \sigma$;*
(ii) *ρ is an order-preserving mapping from partially ordered set $(A, \leqslant_{\mathbf{A}})$ into itself, i.e., for any $a, b \in A$ if $a \leqslant_{\mathbf{A}} b$, then $\rho(a) \leqslant_{\mathbf{A}} \rho(b)$;*
(iii) *$\rho(a) \leqslant_{\mathbf{A}} a$ for any $a \in A$.*

Proof. Let $\mathbf{A} = (A, X, \delta) \in \mathcal{CCAA}$ and g be the unique generator of $\mathbf{A}$. Given an endomorphism ρ of $\mathbf{A}$. Then for any $a \in A$, there exists $u \in X^*$ such that $a = \delta(g, u)$.

(i) Suppose that $\sigma \in E(\mathbf{A})$ and $\rho(g) = \sigma(g)$. Then we have

$$\begin{aligned}
\rho(a) &= \rho\bigl(\delta(g, u)\bigr) \\
&= \delta\bigl(\rho(g), u\bigr) \quad (\textit{since } \rho \textit{ is an endomorphism}) \\
&= \delta\bigl(\sigma(g), u\bigr) \quad \bigl(\textit{since } \rho(g) = \sigma(g)\bigr) \\
&= \sigma\bigl(\delta(g, u)\bigr) \quad (\textit{since } \sigma \textit{ is an endomorphism}) \\
&= \sigma(a).
\end{aligned}$$

This implies that $\rho = \sigma$, as required.

(ii) Suppose that $a, b \in A$ and $a \leqslant_{\mathbf{A}} b$. Then there exists $u \in X^*$ such that $\delta(b, u) = a$. Since $\rho \in E(\mathbf{A})$, we have $\delta(\rho(b), u) = \rho(\delta(b, u)) = \rho(a)$. This is, $\rho(a) \leqslant_{\mathbf{A}} \rho(b)$. Thus, ρ is order-preserving, as required.

(iii) Since g is the generator of $\mathbf{A}$, it is true that for any $a \in A$, there must exist some $u, v \in X^*$ such that $\delta(g, u) = a$ and $\delta(g, v) = \rho(a)$. Thus, we have

$$\begin{aligned}
&\delta(a, u) = \delta\bigl(\delta(g, u), u\bigr) = \delta(g, u) = a; \\
&\delta\bigl(\rho(a), u\bigr) = \rho\bigl(\delta(a, u)\bigr) = \rho(a).
\end{aligned}$$

Then

$$\begin{aligned}
\delta(a, v) &= \delta\bigl(\delta(g, u), v\bigr) \\
&= \delta(g, uv) = \delta(g, vu) \\
&= \delta\bigl(\delta(g, v), u\bigr) \\
&= \delta\bigl(\rho(a), u\bigr) \\
&= \rho(a).
\end{aligned}$$

This implies that $\rho(a) \leqslant_{\mathbf{A}} a$, as required. $\square$

Let $\mathbf{A} = (A, X, \delta)$ be an automaton. For any $x \in X^*$, we denote the set

$$\bigl\{y \in X^* \bigm| (\forall a \in A)\ \delta(a, x) = \delta(a, y)\bigr\}$$

by $\bar{x}$ and denote the set $\{\bar{x} \mid x \in X^*\}$ by $C(\mathbf{A})$. Then $C(\mathbf{A})$ is a monoid under the operation defined by $\bar{x}\bar{y} = \overline{xy}$. It is called the *characteristic monoid* [8] of automaton $\mathbf{A}$.

The following result is shown by Peák in [20].

Lemma 4. *(See [20].) Let $\mathbf{A} = (A, X, \delta)$ be a cyclic commutative automaton. Then we have*

(i) $E(\mathbf{A}) \cong C(\mathbf{A})$;
(ii) $|E(\mathbf{A})| = |A|$.

Also, it is easy to prove that $C(\mathbf{A}) \cong \Lambda(\mathbf{A})$. Thus by Lemma 4, we can show that $E(\mathbf{A}) \cong \Lambda(\mathbf{A})$. In fact, from Proposition 2 we immediately have

Lemma 5. *If $\mathbf{A}$ is an automaton in $\mathcal{CCAA}$, then $E(\mathbf{A}) = \Lambda(\mathbf{A})$ and $E(\mathbf{A})$ is a commutative idempotent monoid.*

Let $\mathbf{A}$ be an automaton in $\mathcal{CCAA}$. We have from the above Lemma 5 that $E(\mathbf{A})$ is a meet semilattice under the natural partial order, $\preccurlyeq$, defined in [9] as follows:

$$\big(\forall \rho, \sigma \in E(\mathbf{A})\big) \quad \rho \preccurlyeq \sigma \quad \Leftrightarrow \quad \rho\sigma = \sigma\rho = \rho.$$

Also, we have

Lemma 6. *If $\mathbf{A} = (A, X, \delta)$ is an automaton in $\mathcal{CCAA}$, then the partially ordered sets $(A, \leqslant_{\mathbf{A}})$ and $(E(\mathbf{A}), \preccurlyeq)$ are order isomorphic, i.e., there exists a bijection θ from A onto $E(\mathbf{A})$ such that both θ and θ^{-1} are order-preserving.*

Proof. Let $\mathbf{A} = (A, X, \delta) \in \mathcal{CCAA}$ and g be the unique generator of $\mathbf{A}$. By Lemma 5, $E(\mathbf{A}) = \Lambda(\mathbf{A})$. Define a mapping θ from A into $E(\mathbf{A})$ as follows:

$$(\forall a \in A) \quad \theta(a) = \lambda_u, \quad \text{where } \lambda_u(g) = a,\ u \in X^*.$$

First, we show that θ is well defined. Noticing that g is a generator, we have that for any $a \in A$, there exists $u \in X^*$ such that $\lambda_u(g) = \delta(g, u) = a$. Moreover, if there exist $\lambda_u, \lambda_v \in \Lambda(\mathbf{A})$ such that $\lambda_u(g) = \lambda_v(g)$, then by Proposition 5(i), we have $\lambda_u = \lambda_v$. Thus, θ is well defined.

Now, we prove that θ is a bijection. Given $a, b \in A$. Suppose that $\theta(a) = \lambda_u$ and $\theta(b) = \lambda_v$ for some $u, v \in X^*$. If $\lambda_u = \lambda_v$, then $a = \lambda_u(g) = \lambda_v(g) = b$. Hence, θ is injective. Since A is finite, we have by Lemma 4(ii) that θ is also surjective. Thus, θ is a bijection.

Next, we prove that θ is order-preserving. Suppose that $a \leqslant_{\mathbf{A}} b$. That is to say, $\delta(b, w) = a$ for some $w \in X^*$. Moreover, we assume that $\theta(a) = \lambda_u$ and $\theta(b) = \lambda_v$. It is clear from Proposition 2 that $\lambda_u\lambda_v = \lambda_v\lambda_u = \lambda_{uv}$. Also, we have

$$\lambda_v(b) = \delta(b, v) = \delta\big(\delta(g, v), v\big) = \delta(g, v) = b.$$

Thus

$$\begin{aligned}
\lambda_{uv}(g) &= \delta(g, uv) = \delta\big(\delta(g, u), v\big) \\
&= \delta(a, v) = \delta\big(\delta(b, w), v\big) \quad \big(\text{since } \delta(b, w) = a\big) \\
&= \delta\big(\delta(b, v), w\big) \quad (\text{since } \mathbf{A} \text{ is commutative}) \\
&= \delta(b, w) \quad \big(\text{since } \delta(b, v) = b\big) \\
&= a = \lambda_u(g).
\end{aligned}$$

Then by Proposition 5(i), we have $\lambda_u\lambda_v = \lambda_v\lambda_u = \lambda_{uv} = \lambda_u$. This implies that $\lambda_u \preccurlyeq \lambda_v$ and hence θ is order-preserving.

Finally, we prove that θ^{-1} is also order-preserving. Given $\lambda_u, \lambda_v \in \Lambda(\mathbf{A})$ such that $\lambda_u \preccurlyeq \lambda_v$. That is to say, $\lambda_u\lambda_v = \lambda_v\lambda_u = \lambda_u$. Assume that $\theta^{-1}(\lambda_u) = a$ and $\theta^{-1}(\lambda_v) = b$. Then we have

$$\begin{aligned}
a &= \lambda_u(g) \quad \big(\text{since } \theta^{-1}(\lambda_u) = a\big) \\
&= \lambda_u\lambda_v(g) \\
&= \lambda_u(b) \quad \big(\text{since } \theta^{-1}(\lambda_v) = b\big) \\
&= \delta(b, u).
\end{aligned}$$

This shows that $a \leqslant_{\mathbf{A}} b$. Hence, θ^{-1} is order-preserving. Therefore, $(A, \leqslant_{\mathbf{A}})$ and $(E(\mathbf{A}), \preccurlyeq)$ are order isomorphic to each other. $\square$

Lemma 7. *Let $\mathbf{A} = (A, X, \delta)$ be an automaton in $\mathcal{CCAA}$ and g the unique generator of $\mathbf{A}$. Then $\rho(g)$ is the maximum fixed point of ρ in $(A, \leqslant_{\mathbf{A}})$ for any $\rho \in E(\mathbf{A})$, i.e., $\rho(g) = \max\{a \in A \mid \rho(a) = a\}$.*

Proof. Let $\mathbf{A} = (A, X, \delta)$ be an automaton in $\mathcal{CCAA}$ and g the unique generator of $\mathbf{A}$. Suppose that $\rho \in E(\mathbf{A})$. Then we have from Lemma 5 that $\rho^2 = \rho$. In particular, $\rho^2(g) = \rho(g)$. That is to say, $\rho(g) \in \{a \in A \mid \rho(a) = a\}$.

For any $b \in A$, there exists $u \in X^*$ such that $\delta(g, u) = b$, since g is a generator. If b is a fixed point of ρ, then $\delta(\rho(g), u) = \rho(\delta(g, u)) = \rho(b) = b$. That is to say, $b \leqslant_{\mathbf{A}} \rho(g)$ and hence $\rho(g) = \max\{a \in A \mid \rho(a) = a\}$. □

Proposition 6. *If $\mathbf{A}$ is an automaton in $\mathcal{CCAA}$, then $E(\mathbf{A}) = \Omega(\mathbf{A})$ holds.*

Proof. Let $\mathbf{A} = (A, X, \delta) \in \mathcal{CCAA}$ and g be the unique generator of $\mathbf{A}$. Since $(E(\mathbf{A}), \preccurlyeq)$ is a meet semilattice, we have from Lemma 6 that $(A, \leqslant_{\mathbf{A}})$ is also a meet semilattice. Thus, $(A, \wedge_{\mathbf{A}})$ is a commutative idempotent monoid and is isomorphic to $E(\mathbf{A})$. It follows from Theorem 1.1.2 in [9] that $(A, \wedge_{\mathbf{A}})$ is also isomorphic to $\Omega(\mathbf{A})$. Then we have $E(\mathbf{A}) \cong \Omega(\mathbf{A})$ and so $|E(\mathbf{A})| = |\Omega(\mathbf{A})|$.

In the following we prove that $E(\mathbf{A}) = \Omega(\mathbf{A})$. Recall the bijection θ from A onto $E(\mathbf{A})$, defined in Lemma 6:

$$(\forall a \in A) \quad \theta(a) = \lambda_u, \quad \text{where } \lambda_u(g) = a,\ u \in X^*.$$

Given $a \in A$ and suppose that $\theta(a) = \lambda_u$. Then we have $\lambda_u^2 = \lambda_u$ and $\lambda_u(a) = \lambda_u^2(g) = \lambda_u(g) = a$. We will show that $\lambda_u = \rho_a$.

First, we prove that $\lambda_u(b) = a \wedge_{\mathbf{A}} b$ holds when $a \leqslant_{\mathbf{A}} b$. If $a \leqslant_{\mathbf{A}} b$, then we have from Proposition 5(ii) that $a = \lambda_u(a) \leqslant_{\mathbf{A}} \lambda_u(b) \leqslant_{\mathbf{A}} \lambda_u(g) = a$. Thus $\lambda_u(b) = a = a \wedge_{\mathbf{A}} b$.

Next, we prove that $\lambda_u(b) = a \wedge_{\mathbf{A}} b$ holds when $b \leqslant_{\mathbf{A}} a$. If $b \leqslant_{\mathbf{A}} a$, then $\delta(a, w) = b$ for some $w \in X^*$. Hence, we have

$$\begin{aligned}
\lambda_u(b) &= \lambda_u\big(\delta(a, w)\big) \\
&= \delta\big(\lambda_u(a), w\big) \quad (\textit{since } \lambda_u \textit{ is an endomorphism}) \\
&= \delta(a, w) \quad \big(\textit{since } \lambda_u(a) = a\big) \\
&= b \\
&= a \wedge_{\mathbf{A}} b.
\end{aligned}$$

Suppose now that a and b are incomparable in $(A, \leqslant_{\mathbf{A}})$. Furthermore, we assume that $a \wedge_{\mathbf{A}} b = d$ and $\lambda_u(b) = d' \neq d$. Then we have from Proposition 5(iii) that $d' \leqslant_{\mathbf{A}} b$. Since $\lambda_u(d') = \lambda_u^2(b) = \lambda_u(b) = d'$, d' is a fixed point of λ_u. Then by Lemma 7, we have $d' \leqslant_{\mathbf{A}} \lambda_u(g) = a$. This implies that d' is a lower bond of a and b. We thus have $d' <_{\mathbf{A}} d$. On the other hand, from $d \leqslant_{\mathbf{A}} a$, we have $\lambda_u(d) = a \wedge_{\mathbf{A}} d = d$ (we have proved it above). Then $\lambda_u(b) = d' <_{\mathbf{A}} d = \lambda_u(d)$, which is a contradiction with the fact that λ_u is order-preserving (Proposition 5(ii)). That is to say, $\lambda_u(b) = a \wedge_{\mathbf{A}} b$ holds when b and a are incomparable in $(A, \leqslant_{\mathbf{A}})$.

Therefore, $\lambda_u = \rho_a$ and hence $E(\mathbf{A}) \subseteq \Omega(\mathbf{A})$. Since $|E(\mathbf{A})| = |\Omega(\mathbf{A})| = |A|$ and A is finite, $\Omega(\mathbf{A}) = E(\mathbf{A})$, as required. □

From Lemma 6 and Proposition 6, we immediately have

Proposition 7. *$\mathcal{CCAA}$ is a subclass of $\mathcal{NCAA}$.*

In order to provide representations of automata in $\mathcal{CCAA}$, we introduce regular S-automata.

Definition 3. Let $(S, \wedge)$ be a finite commutative idempotent monoid. An S-automaton $\mathbf{T} = (T, X, \delta_\varphi)$ is said to be regular if $T = S$ and $S \cong E(\mathbf{T})$.

Let $\mathbf{A} = (A, X, \delta)$ be an automaton in $\mathcal{CCAA}$. From Theorem 1, we know that there exists an S-automaton $\Omega(\mathbf{A}) = (\Omega(\mathbf{A}), X, \delta_\varphi)$ isomorphic to the automaton $\mathbf{A}$, where S is the commutative idempotent monoid $\Omega(\mathbf{A}) \cup \Lambda(\mathbf{A})$ and $\varphi(u) = \lambda_u$ for any $u \in X^*$. Then it is clear that the endomorphism monoid $E(\Omega(\mathbf{A}))$ of the automaton $\Omega(\mathbf{A})$ is isomorphic to $E(\mathbf{A})$.

By Lemma 5 and Proposition 6, we have

$$E(\mathbf{A}) = \Omega(\mathbf{A}) = \Lambda(\mathbf{A}).$$

That is to say, $S = \Omega(\mathbf{A}) = E(\mathbf{A})$ and hence $S \cong E(\Omega(\mathbf{A}))$. Thus, the automaton $\Omega(\mathbf{A})$ is a regular S-automaton. Then from Lemma 2 we immediately have

Theorem 2. *Let $\mathbf{A}$ be an automaton in $\mathcal{CCAA}$ and let S be a semilattice such that $S \cong E(\mathbf{A})$. Then $\mathbf{A}$ is isomorphic to some regular S-automaton.*

4. Cartesian composition of automata

In this section we will study the cartesian composition $\mathbf{A} \circ \mathbf{B}$ of a strongly connected automaton $\mathbf{A}$ and an automaton $\mathbf{B}$ in $\mathcal{CCAA}$. Let $\mathbf{A} = (A, X, \delta)$ and $\mathbf{B} = (B, Y, \gamma)$ be two automata such that $X \cap Y = \emptyset$. The automaton $(A \times B, X \cup Y, \delta \times \gamma)$, denoted by $\mathbf{A} \circ \mathbf{B}$, is called the cartesian composition of $\mathbf{A}$ and $\mathbf{B}$ [5], where

$$\delta \times \gamma\big((a,b),x\big) = \big(\delta(a,x),b\big) \quad \text{for any } (a,b) \in A \times B \text{ and } x \in X,$$
$$\delta \times \gamma\big((a,b),y\big) = \big(a,\gamma(b,y)\big) \quad \text{for any } (a,b) \in A \times B \text{ and } y \in Y.$$

The following result is shown by Dörfler in [5].

Proposition 8. *(See Corollary 1 in [5].) If $\mathbf{A}$ and $\mathbf{B}$ are two connected automata then $E(\mathbf{A} \circ \mathbf{B}) = E(\mathbf{A}) \times E(\mathbf{B})$.*

Let $\mathbf{A} = (A, X, \delta)$ be a strongly connected automaton and $\mathbf{B} = (B, Y, \gamma)$ an automaton in $\mathcal{CCAA}$. Clearly, both $\mathbf{A}$ and $\mathbf{B}$ are connected. Then by Proposition 8, $E(\mathbf{A} \circ \mathbf{B}) = E(\mathbf{A}) \times E(\mathbf{B})$. Furthermore, Fleck proved in [6] that $E(\mathbf{A}) = G(\mathbf{A})$ and $|G(\mathbf{A})|$ divides $|A|$. Also, it is true that the direct product of a group and a semilattice is a Clifford semigroup [21]. Thus by Lemma 4, we immediately have

Corollary 2. *Let $\mathbf{A} = (A, X, \delta)$ be a strongly connected automaton and let $\mathbf{B} = (B, Y, \gamma)$ be an automaton in $\mathcal{CCAA}$. If $X \cap Y = \emptyset$, then the following statements are true*:

(i) $E(\mathbf{A} \circ \mathbf{B}) = G(\mathbf{A}) \times E(\mathbf{B})$;
(ii) $|E(\mathbf{A} \circ \mathbf{B})|$ *divides* $|A| \times |B|$;
(iii) $E(\mathbf{A} \circ \mathbf{B})$ *is a Clifford monoid.*

5. Clifford monoid-type automata

In order to provide a representation of the cartesian composition of a strongly connected automaton and an automaton in $\mathcal{CCAA}$, we introduce Clifford monoid-matrix type automata, which is analogous to the group-matrix type automata introduced by Ito in [14].

Let C be a finite Clifford monoid. Adjoining a zero element, 0, to the set C, we obtain a *Clifford monoid with zero* [9] denoted by C^0. That is to say, for any $c \in C$, $c \cdot 0 = 0 \cdot c = 0$ and $0 \cdot 0 = 0$.

Definition 4. Let C^0 be a finite Clifford monoid with zero. Define a partial operation $+$ on C^0 as follows:

(i) For any $c \in C$, we define $c + 0 = 0 + c = c$ and $0 + 0 = 0$;
(ii) For any $c, d \in C$, $c + d$ is not defined.

We will use the notation $\sum_{i=1}^{n} c_i$ instead of $c_1 + c_2 + \cdots + c_n$. Notice that the sum $\sum_{i=1}^{n} c_i$ is defined only if at most one of c_i, $i = 1, 2, \ldots, n$, is not zero.

Definition 5. Let C be a finite Clifford monoid and n a positive integer. Given an $n \times n$ matrix (c_{pq}) over C^0, where $p, q = 1, 2, \ldots, n$. If for any p, there exists a unique number q such that $c_{pq} \neq 0$, then (c_{pq}) is called a Clifford monoid-matrix of order n on C. Denote the set of all Clifford monoid-matrices of order n on C by $\widetilde{C_n}$.

It is clear that $\widetilde{C_n}$ forms a semigroup under the usual multiplication of matrices.

Definition 6. Let C be a finite Clifford monoid and n a positive integer. Given an n dimension vector (c_p), where $c_p \in C^0$, $p = 1, 2, \ldots, n$. If there exists a unique positive number $i \leqslant n$ such that $c_i \neq 0$, then (c_p) is called a Clifford monoid-vector of order n on C. Denote the set of all Clifford monoid-vectors of order n on C by $\widehat{C_n}$.

For convenience, we always denote a Clifford monoid-vector as $(c)_i$, which means that the i-th component of this vector is c and the others are all equal to zero.

Let $(c_p) = (c)_i \in \widehat{C_n}$ and $(d_{pq}) \in \widetilde{C_n}$. Suppose that the ij entry of (d_{pq}) is h and $h \neq 0$. Then define the following multiplication:

$$(c)_i(d_{pq}) = (c_p)(d_{pq}) = \left(\sum_{k=1}^{n} c_k d_{kp}\right) = (ch)_j.$$

Thus, we have $(c)_i(d_{pq}) \in \widehat{C_n}$.

Definition 7. Let C be a finite Clifford monoid and let n be a positive integer. An automaton $\mathbf{C} = (\widehat{C_n}, X, \delta_\Theta)$ is called a Clifford monoid-matrix type automaton of order n on C, or simply an (n, C)-automaton, if the state transition function δ_Θ is defined by $\delta_\Theta((c)_i, x) = (c)_i\Theta(x)$ for any $(c)_i \in \widehat{C_n}$ and any $x \in X$, where Θ is a mapping from X into $\widetilde{C_n}$.

We always take $\Theta(x) = (\theta_{pq}(x))$ for any $x \in X$, where θ_{pq} is a mapping from X into C^0 and $\theta_{pq}(x)$ is the pq entry of $\Theta(x)$, $p, q = 1, 2, \ldots, n$.

Also, the mapping Θ in the above definition can be extended to a monoid homomorphism from X^* into $\widetilde{C_n}$ as follows: (1) $\Theta(\varepsilon) = (e_{pq})$, $e_{pq} = 0$ if $p \neq q$ and $e_{pp} = e$, where e is the identity of C; (2) $\Theta(xu) = \Theta(x)\Theta(u)$ for any $x \in X$ and any $u \in X^*$.

Let $(S, \wedge)$ be a finite commutative idempotent monoid. Then $(S, \wedge)$ is a Clifford monoid. Thus, a regular S-automaton is a $(1, C)$-automaton, where $C = S$.

The following proposition is analogous to Theorem 1.2.1 in [18].

Proposition 9. *Let $\mathbf{C} = (\widehat{C_n}, X, \delta_\Theta)$ be an (n, C)-automaton. Then C can be embedded into the endomorphism monoid $E(\mathbf{C})$ of the automaton $\mathbf{C}$.*

Proof. Let $\mathbf{C} = (\widehat{C_n}, X, \delta_\Theta)$ be an (n, C)-automaton. For any $c \in C$, define a mapping τ_c from $\widehat{C_n}$ onto itself:

$$\big(\forall (d)_i \in \widehat{C_n}\big) \quad \tau_c\big((d)_i\big) = (cd)_i.$$

We now prove that $\tau_c \in E(\mathbf{C})$. For any $(d)_i \in \widehat{C_n}$ and any $x \in X$, we have

$$\begin{aligned}
\tau_c\big(\delta_\Theta\big((d)_i, x\big)\big) &= \tau_c\big((d)_i\Theta(x)\big) \quad (\textit{by Definition } 7)\\
&= c\big((d)_i\Theta(x)\big)\\
&= (cd)_i\Theta(x)\\
&= \tau_c\big((d)_i\big)\Theta(x)\\
&= \delta_\Theta\big(\tau_c\big((d)_i\big), x\big).
\end{aligned}$$

This implies that $\tau_c \in E(\mathbf{C})$. In the following we prove that the mapping $\phi : c \mapsto \tau_c$ is a homomorphism from C into $E(\mathbf{C})$. For this purpose, put $c \mapsto \tau_c$, $c' \mapsto \tau_{c'}$. Then for any $(d)_i \in \hat{C}$, we have

$$\tau_{cc'}\big((d)_i\big) = \big(cc'd\big)_i = \tau_c\big(\big(c'd\big)_i\big) = \tau_c\big(\tau_{c'}\big((d)_i\big)\big) = \tau_c\tau_{c'}\big((d)_i\big).$$

This means that $cc' \mapsto \tau_c\tau_{c'}$ holds, i.e., ϕ is a homomorphism.

We can easily see that the mapping ϕ is injective. Then C can be embedded into the endomorphism monoid $E(\mathbf{C})$ of the automaton $\mathbf{C}$. □

Definition 8. An (n, C)-automaton $\mathbf{C}$ is said to be regular if $C \cong E(\mathbf{C})$ holds.

6. Representations of automata

In this section we provide a representation of $\mathbf{A} \circ \mathbf{B}$ by regular Clifford monoid matrix-type automaton, where $\mathbf{A}$ is a strongly connected automaton and $\mathbf{B}$ is an automaton in $\mathcal{CCAA}$. This generalizes and extends the representations of strongly connected automata given by Ito in [14].

Lemma 8. *(See Theorem 1.3.1 in [18].) Let $\mathbf{A} = (A, X, \delta)$ be a strongly connected automaton such that $|A| = n|G(\mathbf{A})|$, where n is a positive integer. Then there exists a regular $(n, G(\mathbf{A}))$-automaton $\mathbf{A}' = (\widehat{G(\mathbf{A})}_n, X, \delta_\Psi)$ such that $\mathbf{A}' \cong \mathbf{A}$.*

The following lemma is obvious and its proof is omitted.

Lemma 9. *Let C and C' be two isomorphic Clifford monoids. Then for any (n, C)-automaton $\mathbf{C} = (\widehat{C_n}, X, \delta_\Theta)$, there exists an (n, C')-automaton $\mathbf{C}' = (\widehat{C'_n}, X, \delta_{\Theta'})$ such that $\mathbf{C} \cong \mathbf{C}'$. In addition, if $\mathbf{C}$ is regular, then $\mathbf{C}'$ is also regular.*

Theorem 3. *Let $\mathbf{A} = (A, X, \delta)$ be a strongly connected automaton such that $|A| = n|G(\mathbf{A})|$ and let $\mathbf{B} = (B, Y, \gamma)$ be an automaton in $\mathcal{CCAA}$. Moreover, assume that C is a finite Clifford monoid such that $C \cong E(\mathbf{A} \circ \mathbf{B})$. Then there exists a regular (n, C)-automaton which is isomorphic to $\mathbf{A} \circ \mathbf{B}$.*

Proof. Let $\mathbf{A} = (A, X, \delta)$ be a strongly connected automaton such that $|A| = n|G(\mathbf{A})|$ and let $\mathbf{B} = (B, Y, \gamma)$ be an automaton in $\mathcal{CCAA}$. Moreover, assume that C is a finite Clifford monoid such that $C \cong E(\mathbf{A} \circ \mathbf{B})$.

We know from Lemma 8 that there exists a regular $(n, G(\mathbf{A}))$-automaton $\mathbf{A}' = (\widehat{G(\mathbf{A})}_n, X, \delta_\Psi)$ such that $\mathbf{A} \cong \mathbf{A}'$. Also, it is followed by Theorem 2 that there exists a regular S-automaton $\mathbf{B}' = (S, Y, \gamma_\varphi)$ such that $\mathbf{B} \cong \mathbf{B}'$. Then it is easy to verify that $\mathbf{A} \circ \mathbf{B} \cong \mathbf{A}' \circ \mathbf{B}'$.

On the other hand, we know that $E(\mathbf{B}) \cong E(\mathbf{B}') = S$ (since $\mathbf{B} \cong \mathbf{B}'$ and $\mathbf{B}'$ is regular). Hence, we have from Corollary 2 that $E(\mathbf{A} \circ \mathbf{B}) = G(\mathbf{A}) \times E(\mathbf{B}) \cong G(\mathbf{A}) \times S$. That is to say, $C \cong G(\mathbf{A}) \times S$. Then by Lemma 9, we need only to prove that the automaton $\mathbf{A} \circ \mathbf{B} = (\widehat{G(\mathbf{A})}_n \times S, X \cup Y, \delta_\Psi \times \gamma_\varphi)$ is isomorphic to a regular $(n, G(\mathbf{A}) \times S)$-automaton.

Define a mapping Θ from $X \cup Y$ into $(\widehat{G(\mathbf{A}) \times S})_n$ as follows:

(i)

$$\Theta(x) = \big(\theta_{pq}(x)\big) = \begin{cases} (\psi_{pq}(x), t) & \text{if } \psi_{pq}(x) \neq 0, \\ 0 & \text{if } \psi_{pq}(x) = 0, \end{cases}$$

where $\Psi(x) = (\psi_{pq}(x))$ ([14] or [18]) and t is the identity of the commutative idempotent monoid $(S, \wedge)$;

(ii)

$$\Theta(y) = \big(\theta_{pq}(y)\big) = \begin{cases} (e, \varphi(y)) & \text{if } p = q, \\ 0 & \text{if } p \neq q, \end{cases}$$

where e is the identity of $G(\mathbf{A})$.

Thus, we obtain an $(n, G(\mathbf{A}) \times S)$-automaton

$$\mathbf{C} = \big((\widehat{G(\mathbf{A}) \times S})_n, X \cup Y, \sigma_\Theta\big).$$

Next, we prove that $\mathbf{A}' \circ \mathbf{B}' \cong \mathbf{C}$. To this end, we define a mapping ρ from $\widehat{G(\mathbf{A})}_n \times S$ into $(\widehat{G(\mathbf{A}) \times S})_n$ as follows:

> $\rho : ((g)_i, s) \mapsto (g, s)_i$, where $(g)_i$ is a group-vector [18] of order n on $G(\mathbf{A})$ whose i-th component is g and others are all equal to zero; $(g, s)_i$ is a Clifford monoid-vector of order n on $G(\mathbf{A}) \times S$ whose i-th component is (g, s) and others are all equal to zero.

Obviously, ρ is a bijection.

We now prove that ρ is a homomorphism, i.e.,

$$\rho\big(\delta_\Psi \times \gamma_\varphi\big(\big((g)_i, s\big), x\big)\big) = \sigma_\Theta\big(\rho\big((g)_i, s\big), x\big), \tag{1}$$

$$\rho\big(\delta_\Psi \times \gamma_\varphi\big(\big((g)_i, s\big), y\big)\big) = \sigma_\Theta\big(\rho\big((g)_i, s\big), y\big) \tag{2}$$

hold for any $x \in X$, $y \in Y$ and any $((g)_i, s) \in \widehat{G(\mathbf{A})}_n \times S$.

Given $(g)_i \in \widehat{G(\mathbf{A})}_n$ and $x \in X$. We know from Definition 1.2.2 in [18] that there exists a unique $j \leqslant n$ such that $\psi_{ij}(x) \neq 0$. Suppose that $\psi_{ij}(x) = h$. Then $\theta_{ij}(x) = (h, t)$. Thus, we have

$$\begin{aligned} \rho\big(\delta_\Psi \times \gamma_\varphi\big(\big((g)_i, s\big), x\big)\big) &= \rho\big(\delta_\Psi\big((g)_i, x\big), s\big) \\ &= \rho\big((g)_i \Psi(x), s\big) \quad (\textit{by Definition } 1.2.4 \textit{ in } [18]) \\ &= \rho\big((gh)_j, s\big) \quad \big(\textit{since } \psi_{ij}(x) = h\big) \\ &= (gh, s)_j. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma_\Theta\big(\rho\big((g)_i, s\big), x\big) &= \sigma_\Theta\big((g, s)_i, x\big) \\ &= (g, s)_i \Theta(x) \quad (\textit{by Definition } 7) \\ &= (gh, s \wedge t)_j \quad \big(\textit{since } \theta_{ij}(x) = (h, t)\big) \\ &= (gh, s)_j. \end{aligned}$$

Then Eq. (1) holds for any $((g)_i, s) \in \widehat{G(\mathbf{A})}_n \times S$ and any $x \in X$.

Also, for any $((g)_i, s) \in \widehat{G(\mathbf{A})_n} \times S$ and any $y \in Y$, we have

$$\begin{aligned}\rho\big(\delta_\Psi \times \gamma_\varphi\big(\big((g)_i, s\big), y\big)\big) &= \rho\big((g)_i, \gamma_\varphi(s, y)\big)\\ &= \rho\big((g)_i, s \wedge \varphi(y)\big) \quad (\textit{by Definition } 2)\\ &= \big(g, s \wedge \varphi(y)\big)_i\end{aligned}$$

and

$$\begin{aligned}\sigma_\Theta\big(\rho\big((g_i), s\big), y\big) &= \sigma_\Theta\big((g, s)_i, y\big)\\ &= (g, s)_i \Theta(y) \quad (\textit{by Definition } 7)\\ &= \big(ge, s \wedge \varphi(y)\big)_i \quad \big(\textit{since } \theta_{ii}(y) = \big(e, \varphi(y)\big)\big)\\ &= \big(g, s \wedge \varphi(y)\big)_i.\end{aligned}$$

Then Eq. (2) holds for any $((g)_i, s) \in \widehat{G(\mathbf{A})_n} \times S$ and any $y \in Y$.

Therefore, $\mathbf{A}' \circ \mathbf{B}' \cong \mathbf{C}$. Moreover,

$$E(\mathbf{C}) \cong E\big(\mathbf{A}' \circ \mathbf{B}'\big) \cong E(\mathbf{A} \circ \mathbf{B}) \cong G(\mathbf{A}) \times S.$$

Then $\mathbf{C}$ is regular. □

Theorem 3 is a generalization of Theorem 2 and Theorem 1.3.1 in [18] since both a semilattice and a group are Clifford semigroups.

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